Physics 316, Final exam December 6, 2005 name: .....

Work the problems below, explaining your reasoning. You may consult notes that you wrote yourself.

1. Mirage When a light ray moves through an inhomogeneous medium, it follows the path of least transit time. We suppose that the speed of light c in some material varies linearly with height y: $c(y) = a + by$. We look for a light path $y(x)$ between two points separated by distance X with $y(0) = y(X) = 0$. The propagation time T may be written $T = \int dx/v_x$, where $v_x = c(y)\cos\theta = c(y)/\sqrt{1+\dot{y}^2}$, where θ is the angle with the x axis and $\dot{y} \equiv dy/dx$.
a) Find a function of y and $\dot{y}$ that is independent of position along the path.
b) Find the maximum height $y_{\max} = y(X/2)$.

Solution: a) 4 points. Finding the path of least time is a form of Hamiltons principle with Lagrangian

$$\mathcal{L} = 1/v_x = \frac{\sqrt{1+\dot{y}^2}}{a+by}$$

Since $\mathcal{L}$ does not depend on x, the value of the Hamiltonian $\mathcal{H} = \dot{y}\ \partial\mathcal{L}/\partial\dot{y} - \mathcal{L}$ does not change with x.

$$\mathcal{H} = \frac{\dot{y}^2}{(a+by)\sqrt{\dots}} - \frac{1}{a+by}\sqrt{\dots} = \frac{1}{(a+by)\sqrt{\dots}}\left[\dot{y}^2 - (1+\dot{y}^2)\right] = \frac{-1}{(a+by)\sqrt{\dots}}$$

b) 6 points. We may solve for $\dot{y}$ and thence determine X as a function of y. We denote the fixed value of $\mathcal{H}$ as E. Evidently,

$$\dot{y} = \left[\left(\frac{1}{E(a+by)}\right)^2 - 1\right]^{1/2} = \frac{(1 - E^2(a+by)^2)^{1/2}}{E(a+by)}$$

Now, we let $x = X/2$ so that $y = y_m$:

$$X/2 = \int_0^{y_m} dy/\dot{y} = \int dy \frac{E(a+by)}{(1-E^2(a+by)^2)^{1/2}}$$

We let $E(a+by) \equiv u$. We also notice that the endpoint y_m of the integration must be where $\dot{y} = 0$. This determines E.

$$X/2 = \frac{1}{Eb}\int_{u_0}^{u_m} du\ u\ ({u_m}^2 - u^2)^{-1/2} = \frac{1}{2\ Eb}\int_{v_0}^{v_m} dv(v_m - v)^{-1/2}$$

where $v \equiv u^2$. Now letting $w \equiv v_m - v$

$$X = \frac{1}{Eb}\int_{w_m}^{w_0} dw\ w^{-1/2}$$

$$EbX = 2\left({w_0}^{1/2} - {w_m}^{1/2}\right).$$

$w_m = 0$, $w_0 = {u_m}^2 - {u_0}^2 = (E(a + by_m))^2 - (Ea)^2$, so our equation reads

$$EbX = 2\left((E(a + by_m))^2 - (Ea)^2\right)^{1/2}.$$

We divide out Eb and denote the length $a/b \equiv \lambda$. Then

$$\frac{1}{2}X = \left((\lambda + y_m)^2 - \lambda^2\right)^{1/2}.$$

Or, defining $\tilde{X} \equiv X/\lambda$; $\tilde{y}_m \equiv y_m/\lambda$,

$$\frac{1}{2}\tilde{X} = \left((1 + \tilde{y}_m)^2 - 1\right)^{1/2}.$$

$$\tilde{y}_m = ((\frac{1}{2}\tilde{X})^2 + 1)^{1/2} - 1.$$

For small gradent b, $\tilde{X} \ll 1$ and $\tilde{y}_m \to \tilde{X}^2/8$. For large gradient or large $\tilde{X}$, $\tilde{y}_m \to \frac{1}{2}\tilde{X}$; the height is half the horizontal distance.

2. Rod gizmo A particle of mass m is attached to the middle of a massless rod of length R. The ends of this rod move on a cylinder also of radius R and of infinite length. The z direction points along the axis of the cylinder. There is no friction, gravity or other external force. To describe this system we use the z coordinate of the midpoint of the rod, the azimuthal angle ϕ of this point and the angle θ that the rod makes with the cylinder axis. We consider initial conditions where $\theta \ll 1$.

a) Find the Lagrangian $\mathcal{L}$ of the system for small θ up to order θ^2.
b) Find the equation of motion for z.
c) We notice that the transformation $\phi \to \phi + c$ where c is a constant leaves the Lagrangian invariant. According to Noether's theorem, this implies a constant of the motion. What is the functional form of this constant in terms of $z, \phi, \theta, \dot{z}, \dot{\phi}$, and $\dot{\theta}$.
d) The system initially has $\dot{\phi} \equiv \Omega$, $\dot{\theta} = 0$ $\dot{z} = 0$ and $0 < \theta \ll 1$. Then θ increases with time. When $\theta \gg \theta_0$ but $\theta \ll 1$, θ has the form At^b, where A, and b. are constants. Find A and b.

Solution: a) 4 points. Co-ordinates of the mass are r, ϕ, z. $\mathcal{L} = T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2)$. We must express r in terms of the rod angle θ. looking down the axis, the horizontal projection of the rod has length $R_\perp = R\sin\theta$. Looking at the right triangle formed by the mass, a contact point and the axis, we infer $r^2 = R^2 - (\frac{1}{2}R_\perp)^2$, or $r = (R^2 - \frac{1}{4}R^2\sin^2\theta)^{1/2} \simeq R(1 - \frac{1}{8}\theta^2)$. $\dot{r} = \frac{1}{4}R\,\theta\dot{\theta}$. Thus

$$\mathcal{L} = \frac{1}{2}m\left(\frac{1}{16}R^2\,\theta^2\dot{\theta}^2 + R^2(1 - \frac{1}{8}\theta^2)^2\dot{\phi}^2 + \dot{z}^2\right)$$

$$\simeq \frac{1}{2}mR^2\left(\frac{1}{16}\,\theta^2\dot{\theta}^2 + (1 - \frac{1}{4}\theta^2)\dot{\phi}^2\right) + \frac{1}{2}m\dot{z}^2$$

From here on I set $m = R = 1$ for simplicity until the end.

b) 1 point. The motion in z is completely decoupled from the other co-ordinates. It's Lagrangian simply says that $\ddot{z} = 0$, *i.e.,* $\dot{z}$ is a constant.

c) 2 points. The Lagrangian is independent of the co-ordinate ϕ, so $\partial\mathcal{L}/\partial\dot{\phi}$ is conserved. This quantity is the angular momentum $p_\phi = (1 - \frac{1}{4}\theta^2)\dot{\phi}$.

d) 5 points. To find the motion in the remaining co-ordinate θ we can use its Lagrange equation or the conservation of $\mathcal{H} = \mathcal{L}$. We denote this constant, without the $\dot{z}$ part, as E. The initial conditions imply $p_\phi = (1 - \frac{1}{4}{\theta_0}^2)\Omega$. We may express $\dot{\phi}$ in terms of p_ϕ: $\dot{\phi} \simeq (1 + \frac{1}{4}\theta^2)p_\phi$.

$$E = \frac{1}{32}\ \theta^2\dot{\theta}^2 + \frac{1}{2}(1 - \frac{1}{4}\theta^2)\dot{\phi}^2 = \frac{1}{32}\ \theta^2\dot{\theta}^2 + \frac{1}{2}(1 + \frac{1}{4}\theta^2)\ {p_\phi}^2$$

The initial value of E is $\frac{1}{2}(1 - \frac{1}{4}{\theta_0}^2)\Omega^2$. This energy has an odd form since there is a θ^2 factor in the kinetic energy. the "mass" is proportional to θ. This is because the outward velocity of the mass for a given $\dot{\theta}$ goes to zero as $\theta \to 0$.

We may find $\theta(t)$ by expressing $\dot{\theta}$ in terms of θ and the constants E and p_ϕ:

$$\dot{\theta}^2 = \frac{32}{\theta^2}(E - \frac{1}{2}(1 + \frac{1}{4}\theta^2)\ {p_\phi}^2)$$

$$t = \int d\theta/\dot{\theta} = \int d\theta\ \theta/4 \left(2E - (1 + \frac{1}{4}\theta^2){p_\phi}^2\right)^{-1/2}$$

Defining $u \equiv \theta^2$, we find

$$t = \frac{1}{2}\int_{\theta_0^2}^{\theta^2} du \left([2E - p_\phi^2] - [\frac{1}{4}{p_\phi}^2]\ u\right)^{-1/2}$$

Denoting the first [..] as A and the second [...] as B, we infer

$$t = \frac{1}{B}\left[(A - B\ u)^{1/2}\right]_{\theta_0^2}^{\theta^2}$$

The lower limit doesn't contribute, since we know $\dot{\theta} = 0$ when $\theta = \theta_0$. Solving for θ^2

$$(B\ t)^2 = A - B\theta^2; \quad \theta = \pm\left(A/B - B\ t^2\right)^{1/2}$$

The angle θ must initially decrease, since the kinetic energy can only increase from its initial value of 0. θ accelerates inward and reaches an infinite $\dot{\theta}$ as it crosses $\theta = 0$. After that, the requirement of constant E means that $\dot{\theta}$ must decrease in magnitude. When $\theta = -\theta_0$. $\dot{\theta}$ must return to its initial value of 0, and the situation is the same as at the start. Thus θ executes oscillatory motion but with a strange time dependence.

The question asks how θ depends on time when $\theta \gg \theta_0$. Our solution that gave this behavior seems to have had an error in it, since it is clear that θ never increases beyond θ_0. I apologize!

3. Canonical generator A particle moves in a plane under the action of an attractive central power-law force with potential $U(r) = k/r^n$. We describe the motion using polar co-ordinates r and θ. The kinetic energy $T = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2$. Thus the Hamiltonian $\mathcal{H}$ in terms of

r, θ and their conjugate momenta p_r, and p_θ has the usual form $\mathcal{H} = \frac{1}{2}{p_r}^2 + \frac{1}{2}{p_\theta}^2/r^2 - U(r)$. We define a second Hamiltonian $\mathcal{D} \equiv r p_r$. We denote its time variable as s.

a) Express dr/ds as a function of phase space co-ordinates; do the same for $\frac{d}{ds}p_r$, $\frac{d}{ds}\theta$, and $\frac{d}{ds}p_\theta$.

b) Using the result of a), find $r(s)$, $p_r(s)$ $\theta(s)$, $p_\theta(s)$ in terms of $r(0), p_r(0), \theta(0), p_\theta(0)$.

c) What is the image of an elliptical Kepler orbit with maximum distance R and eccentricity ϵ under this transformation when $s = 1$?

d) Find the value of $\mathcal{D}(s)$ in terms of $\mathcal{D}(0)$.

e) Find a value of the exponent n such that $\frac{d}{ds}\mathcal{H}$ has the form $a\ \mathcal{H}$, and give the value of the numerical constant a.

f) For the n found in e) find an expression for $\frac{d}{dt}\mathcal{D}$ in terms of $\mathcal{H}$. How does $\mathcal{D}$ vary in time?

solution: a) 2 points. As told in Chapter 8, the phase space function $\mathcal{D}$, viewed as a Hamiltonian, generates motion via Hamilton's equations:

$$\frac{d}{ds}r = \frac{\partial \mathcal{D}}{\partial p_r} = r; \quad \frac{d}{ds}p_r = -\frac{\partial \mathcal{D}}{\partial r} = -p_r; \quad \frac{d}{ds}\theta = \frac{\partial \mathcal{D}}{\partial \theta} = 0; \quad \frac{d}{ds}p_\theta = 0$$

b) 2 points. $r(s) = r(0)e^s$; $p_r(s) = p_r(0)e^{-s}$; $\theta(s) = \theta(0); p_\theta(s) = p_\theta(0)$. The transformation dilates co-ordinate space by a factor e^s and contracts momentum space by the same factor, with no change of the angles.

c) 2 points. The orbit is dilated as described in b). R is increased by a factor e^s; Since the dilation is uniform; it does not change any ratio of lengths, so the eccentricity ϵ is unchanged.

d) 1 point. $(d\ /ds)\mathcal{D} = \{\mathcal{D}, \mathcal{D}\} = 0$, confirming the fact that any hamiltonian flow preserves the value of the Hamiltonian function if it doesn't explicitly depend on time.

e) 3 points. $(d\ /ds)\mathcal{H} = \{\mathcal{H}, \mathcal{D}\}$. Thus

$$(d\ /ds)\mathcal{H} = \frac{\partial \mathcal{H}}{\partial r}\frac{\partial \mathcal{D}}{\partial p_r} - \frac{\partial \mathcal{D}}{\partial r}\frac{\partial \mathcal{H}}{\partial r}$$

The partial derivatives involving θ and p_θ vanish since $\mathcal{D}$ is independent of these. Now

$$\frac{\partial \mathcal{H}}{\partial r} = -2p_\theta^2/r^3 - nk/r^{n+1}; \quad \frac{\partial \mathcal{H}}{\partial p_r} = 2p_r.$$

Combining,

$$(d\ /ds)\mathcal{H} = (-2p_\theta^2/r^3 - nk/r^{n+1})(r) - (p_r)(2p_r) = -2p_r^2 - 2p_\theta^2/r^2 - nk/r^n$$

Alternatively, one may directly take $(d\ /ds)\mathcal{H}(r(s), p_r(s), \theta, p_\theta)$. This looks just like the Hamiltonian provided $n = 2$. Then the right hand side is just $-2\mathcal{H}$, so $a = -2$.

f) 2 points. $d\ /dt\ \mathcal{D} = \{\mathcal{D}, \mathcal{H}\} = -\{\mathcal{H}, \mathcal{D}\} = -(d\ /ds)\mathcal{H} = +2\mathcal{H}$. Since $\mathcal{H}$ is constant in time, one may directly integrate to obtain $\mathcal{D}(t) = \mathcal{D}(0)(2\mathcal{H})t$.